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PROCEEDINGS

OF

THE ROYAL IRISH ACADEMY.

1846. No. 52.

January 12, 1846.

GEORGE PETRIE, Esq, Vice-President, in the Chair.

The Earl of Enniskillen, William Lloyd, M.D., The Ven. Henry Cotton, D.C.L., Archdeacon of Cashel, Rev. John Connell, John C. Deane, Rickard Deasy, Conyngham Ellis, Arthur Nugent, Edward King Tenison, John Tyrrell, and T. Jolliffe Tuffnell, Esquires, were elected Members of the Academy.

RESOLVED,—" That the recommendation of the Council, to place £50 11*s*. at their disposal, for the purchase of a convoluted Gold Bracelet, be adopted."

The Rev. Professor Graves read a paper on the discussion of Algebraic Curves and Surfaces, with special references to the theory of Curves of the third degree.

The general equation of an algebraic curve or surface of the n^{th} degree being written in the form

$$u_n + u_{n-1} + u_{n-2} + \ldots\ldots + u_2 + u_1 + u_0 = 0, \qquad (1)$$

where u_n stands for a homogeneous function of the n^{th} degree of the rectilinear coordinates, any equation such as

$$A_m u_m + A_{m-1} u_{m-1} + A_{m-2} u_{m-2} + \ldots + A_2 u_2 + A_1 u_1 + A_0 u_0 = 0 \; (2)$$

in which m is not greater than n, and A_0, A_1, A_2, ... A_m are constants, represents a curve or surface derived from the former by means of the following geometrical construction.

Draw from the origin O any right line meeting the surface (1) in the n points $N_1, N_2, N_3, \ldots\ldots N_n$; and assume on it m points $M_1, M_2, M_3, \ldots\ldots M_m$, so as to satisfy the m conditions

$$A_0 \Sigma\left(\frac{1}{OM_1}\right) = A_1 \Sigma\left(\frac{1}{ON_1}\right)$$

$$A_0 \Sigma\left(\frac{1}{OM_1 . OM_2}\right) = A_2 \Sigma\left(\frac{1}{ON_1 . ON_2}\right)$$

$$A_0 \Sigma\left(\frac{1}{OM_1 . OM_2 . OM_3}\right) = A_3 \Sigma\left(\frac{1}{ON_1 . ON_2 . ON_3}\right)$$

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$$A_0 \left(\frac{1}{OM_1 . OM_2 . OM_3 \ldots\ldots OM_m}\right) = A_m \Sigma\left(\frac{1}{ON_1 . ON_2 . ON_3 \ldots\ldots ON_m}\right);$$

Then the points $M_1, M_2, M_3, \ldots\ldots M_m$ will lie on the surface (2).

If we suppose now that the coefficients $A_0, A_1, A_2, \ldots A_m$ are formed according to any assumed law; for instance, if they are given functions of m and n, we shall have, as m takes different integer values from 1 up to n, a series of curves or surfaces, derived from the original one, and related in a particular manner to it and to each other.

By making the coefficients $A_0, A_1, A_2, \ldots A_m$, all equal to unity, we form a series of curves or surfaces most easily derived from any given one; and the consideration of them suggests some interesting results.

The problem of drawing a tangent geometrically at a given point on a curve of the third degree has been elegantly solved by M. Poncelet; but we are now in a condition to solve it generally for any algebraic curve.

Having drawn any right line from the given point O, meeting the curve in $n-1$ other points $N_1 N_2 N_3 \ldots . N_{n-1}$, let us assume on it a point M such that

$$\frac{1}{OM} = \Sigma\left(\frac{1}{ON_1}\right), \qquad (3)$$

the locus of the point M will be a conic having a three point osculation with the curve at O. The tangent and osculating circle of this conic belong therefore likewise to the given curve. So also, in the case of a point O given on any algebraic surface, the surface of the second order, which is the locus of a point M determined in the same way, will have its lines of greatest and least curvature coincident with those of the given surface at O. All this is obvious, since if

$$u_n + u_{n-1} + \dots + u_2 + u_1 = 0 \qquad (4)$$

be the equation of the given curve or surface, referred to axes passing through the given point O,

$$u_2 + u_1 = 0 \qquad (5)$$

will be the equation of the curve or surface of the second order, constructed in the manner described above.

Let us suppose $n = 3$, and (4) to be the equation of a plane curve of the third degree; its intersections with the conic (5)* determine the three right lines represented by the equation

$$u_3 = 0$$

which are obviously parallel to the asymptots. The directions of the asymptots being thus ascertained, we may readily determine their actual position. For this purpose draw tangents to the curve of the third order at the extremities of any one of the chords common to it and the conic; they will meet the curve in two points; and the line joining these points will cut the curve in the point through which the asymptot parallel to that chord passes.

Of course the results here stated are subject to modifications when O is a singular point.

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* As we are accustomed to regard the curve as generated by the motion of the tangent, whose equation is $u_1 = 0$, it seems natural to extend this conception, and to call the conic $u_2 + u_1 = 0$ *the generating conic:* the curve $u_3 + u_2 + u_1 = 0$ *the generating curve of the third degree*, and so on.

If we draw from o two right lines parallel to the asymptots of the generating conic; in virtue of the equation (3), they will each meet the curve of the third degree in two points equidistant from o. It follows, therefore, that the two rectilinear diameters of the curve, conjugate to these right lines, must pass through o. Hence we know how to construct these diameters. Again, as there are plainly but two diameters passing through o, the curve enveloped by all the diameters of the curve must be of the *second class,* that is to say, it must be a conic section.

If o be a conjugate point, the conic and its asymptots are imaginary ; consequently o must lie within the envelope.

If o be a cusp, the conic degenerates into two coincident right lines. The diameters of the curve conjugate to them must, therefore, be coincident likewise. Hence o lies on the envelope.

If o be a double point, the conic is replaced by two intersecting right lines; and it is easy to see that, as we can draw from o two diameters of the curve, conjugate to the two systems of respectively parallel chords, o must lie outside the envelope.

The preceding theorems, relative to the envelope of the rectilinear diameters of a curve of the third order, and to the positions of singular points of the curve with respect to it, are due to Professor Plücker of Bonn, who obtained them by analytical methods. They are here presented as consequences, derived by purely geometrical considerations from the properties of the generating conic ; in order to illustrate the importance of the place which it holds in the theory of curves of the third order.

Perhaps the most interesting class of *derived* curves or surfaces is that of the *Successive Polars* of a given curve or surface of the n^{th} order. The geometric mode of generating them is as follows:

Draw from any point o a right line meeting the surface in n points, N_1, N_2, $N_3 \ldots\ldots\ldots N_n$, and assume on it m points,

M_1, M_2, $\mathrm{M}_3 \ldots\ldots \mathrm{M}_m$, such that

$$\Sigma\left(\frac{1}{\mathrm{OM}_1}\right) = \frac{m}{n}\Sigma\left(\frac{1}{\mathrm{ON}_1}\right)$$

$$\Sigma\left(\frac{1}{\mathrm{OM}_1.\mathrm{OM}_2}\right) = \frac{m(m-1)}{n(n-1)}\,\Sigma\left(\frac{1}{\mathrm{ON}_1.\mathrm{ON}_2}\right),$$

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$$\frac{1}{\mathrm{OM}_1.\mathrm{OM}_2\ldots\ldots\mathrm{OM}_m} = \frac{m(m-1)\ldots\ldots2.1}{n(n-1)\ldots(n-m+1)}\Sigma\left(\frac{1}{\mathrm{ON}_1.\mathrm{ON}_2\ldots\ldots\mathrm{ON}_m}\right);$$

the curve or surface which is the locus of the points M_1, M_2, $\mathrm{M}_3 \ldots\ldots \mathrm{M}_m$ will be the m^{th} *polar* of the point o with relation to the given curve or surface. It is plain that, so far as regards this method of geometrical generation, the polars are *successive:* that is to say, the $(m+1)^{th}$ polar is derived from the m^{th} in the same manner as the m^{th} from the $(m-1)^{th}$. In every case the products of the distances OM_1, $\mathrm{OM}_2 \ldots \mathrm{OM}_m$, taken r by r, have the same harmonic mean as the products of ON_1, $\mathrm{ON}_2 \ldots \mathrm{ON}_n$, also taken r by r : and this for all integer values of r from unity to m inclusive.

It follows from what has been already said, that if

$$\mathrm{U}_n + \mathrm{U}_{n-1} + \ldots + \mathrm{U}_2 + \mathrm{U}_1 + \mathrm{U}_0 = \mathrm{o}$$

be the equation of the given curve or surface, referred to rectilinear axes passing through o,

$$\frac{m(m-1)\ldots2.1}{n(n-1)\ldots(n-m+1)}\,\mathrm{U}_m + \frac{m(m-1)\ldots2}{n(n-1)\ldots(n-m+2)}\,\mathrm{U}_{m-1} +$$

$$\ldots + \frac{m(m-1)}{n(n-1)}\,\mathrm{U}_2 + \frac{m}{n}\mathrm{U} + \mathrm{U}_1 = \mathrm{o}$$

will be the equation of the m^{th} polar of the point o, with relation to the given curve or surface. By making m, in this formula, successively equal to the integer numbers from 1 up to $n-1$ inclusive, we obtain the equations of all the *successive polars* of the origin. For instance, the equation of the first polar is

$$\mathrm{U}_1 + n\mathrm{U}_0 = \mathrm{o}$$

which involves in it the celebrated theorem of Cotes, so successfully used by Maclaurin. And the equation of the $(n-1)^{th}$ polar is

$$\mathrm{u}_{n-1} + 2\mathrm{u}_{n-2} + 3\mathrm{u}_{n-3} + \ldots + (n-1)\mathrm{u}_1 + n\mathrm{u}_0 = \mathrm{o},$$

an equation which we recognize as belonging to the curve or surface of the $(n-1)^{th}$ order, which passes through the points of contact of all the tangents drawn from the origin to the given curve or surface. It is by this geometrical property that M. Bobillier, who first directed attention to these successive polars,* has chosen to characterise them.

By substituting $\frac{1}{x}, \frac{y}{x}, \frac{z}{x}$, for x, y, z, in the general equation of the surface, and in the equations of its polars, Professor Graves shews that, when the point o recedes to an infinite distance, the whole series of successive polars become diametral lines or surfaces of the different orders belonging to the given curve or surface. This had, in fact, been observed, in the case of the first polar, by M. Poncelet, who has shewn that the theorem of Cotes is an extension of Newton's proposition relative to the rectilinear diameters of plane curves.†

This theory of polars enables us to give a geometrical construction of the problem, "From a given point in its plane to draw all the possible tangents to a curve of the third order."

We have only to construct the second polar of the given point, which will be a conic section, and its intersection with the curve will give the points of contact. Here we see the advantage of adopting the geometrical definition of polars employed in this paper.

In connexion with the present subject, Professor Graves announced an extension of a theorem deduced by Maclaurin

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* See a Series of Articles contained in the eighteenth and nineteenth volumes of Gergonne's *Annales de Mathématique*.

† See Chasles' *Histoire de Geometrie*, p. 147.

from the theorem of Cotes. Maclaurin's theorem is: "If any transversal be drawn through a fixed point O, in the plane of a curve of the n^{th} order, so as to meet the curve in n points, the tangents drawn at these points will cut off upon any fixed right line passing through O, segments OT_1, $OT_2 \ldots OT_n$, the sum of whose reciprocals is constant."

The following is a more general theorem for plane curves: "*If any two transversals be drawn through a fixed point* O, *in the plane of a curve of the* n^{th} *order, so that one meets the curve in* n *points* M_1, $M_2 \ldots M_n$, *and the other in* n *points* M'_1, $M'_2, \ldots M'_n$; *the right lines* $M_1M'_1$, $M_2M'_2, \ldots M_nM'_n$, *will cut off, upon any fixed right line passing through* O, *segments* OT_1, $OT_2, \ldots OT_n$, *the sum of whose reciprocals is constant.*"

And we have an analogous one for surfaces: "*If any three transversals be drawn through a fixed point* O, *the first meeting a surface of the* n^{th} *order in* n *points,* M_1, $M_2, \ldots M_n$; *the second meeting it in* M'_1, $M'_2, \ldots M'_n$; *the third in* M''_1, $M''_2, \ldots M_n''$; *the planes* $M_1M'_1M''_2$, $M_2M'_2M''_2, \ldots M_nM'_nM''_n$ *will cut off, upon any fixed right line passing through* O, *segments, the sum of whose reciprocals is constant.*

The Rev. Dr. Drummond read a paper on the authorship of the poem entitled "The Exile of Erin."

This subject was taken up in consequence of a provincial newspaper having been sent to Sir William Hamilton, accompanied by a letter setting forth the claims of George Nugent Reynolds, Esq., to the authorship of that poem, and requesting that the matter might be brought under the consideration of the Royal Irish Academy. This task was at first declined, as unworthy of serious attention; but the claims of Reynolds continuing to be urged in several publications, and Thomas Campbell, the reputed author, represented as a plagiarist, Dr. Drummond, lest the silence of the Academy should be construed into an admission of the validity of Reynolds'